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Weighted Estimates for Singular Differential Operators in Functional Spaces

ABSTRACT
of dissertation for the degree of Doctor of Philosophy (PhD) in specialty 6D060100 – Mathematics

Relevance of the research. Differential operators, as mathematical models, are used for in-depth analysis in many fields of scientific research, such as mathematical physics, the theory of stochastic processes, biology, financial mathematics, and others. For example, applications of differential operators in mechanics and engineering describe oscillations in viscoelastic media and the dynamics of systems with resistance. In biology, these may include models of population growth and spread with noise (outliers, migrations) and ecological models with spatially varying parameters.

Contemporary research directions in differential operators include, as an integral part, problems associated with differential operators (equations) with various types of singularities of variable coefficients. These singularities include, in particular, conditions of non-smoothness, unboundedness, rapidly growing oscillations, and the sign-alternating property of variable coefficients as they approach the boundary points of their domain:

- Problems related to boundedness, closedness (for unbounded operators), and invertibility. Here, an important role is played by the choice (or construction of new) functional spaces in which the operators under study are considered. Such problems were addressed in the works of W.N. Everitt, M. Giertz, K.Kh. Boimatov, M.O. Otelbaev, A.A. Shkalikov, M. Hieber, L. Lorenzi, A. Rhandi, R. Schnaubelt, K.N. Ospanov, L.K. Kusainova, and others;
- Weighted estimates of solutions to differential equations, which have been extensively studied in the works of many authors, in particular M.O. Otelbaev, R.O. Oinarov, G. Metafune, D. Pallara, J. Prüss, K.N. Ospanov, S. Monsurrò, M. Transirico, K.A. Mirzoev, and others;
- Problems related to the choice of numerical methods for solving the corresponding differential equations. In the language of functional analysis, this is expressed as: the choice of a method for approximating solutions by finite-dimensional objects; the specification of the approximation measure; and the estimation of the approximation accuracy at the n-th iterative step. Such problems were previously addressed by P.I. Lizorkin, M.O. Otelbaev, and are currently being addressed by M.S. Aitenova, L.K. Kusainova, M. Muratbekov, N.K. Makina, E.Ya. Nazirova, Ya.T. Sultanayev, and others.

Aim of the work. The aim of the dissertation is to study certain types of singular ordinary differential operators of the second order on a semi-axis with coefficients unbounded in any neighborhood of infinity.

Research objectives.

1. Study of a non-self-adjoint ordinary differential operator (ODO) with unbounded and possibly sign-alternating coefficients in any neighborhood of $+\infty$. Description of conditions under which a well-defined closed invertible extension exists. Estimates of the best linear approximations for differential equations corresponding to the operator under study.
2. Study of a second-order ODO with coefficients satisfying Triebel-type conditions at $+\infty$. Functional description and estimates of the best linear approximations for solutions of the corresponding differential equations.

3. Derivation of functional coercive estimates for the ODO, namely, in terms of multipliers acting in the solution spaces.

Research methods. The main research method is the method of establishing local relationships between the coefficients of the operators under study on characteristic intervals, which is a development of M.O. Otelbaev's "running averages" method. The work also employs methods of functional analysis, the theory of differential operators, and approximation theory in functional spaces.

Scientific novelty. The following new results were obtained for second-order differential operators:

1. Conditions for the existence of a closed invertible extension of a singular second-order differential operator with locally summable and sign-alternating coefficients in weighted functional spaces have been obtained.

2. Coercive estimates for singular differential operators in Sobolev spaces, Lebesgue spaces, and spaces of pointwise multipliers have been obtained.

3. Theorems on the separability of the Sturm–Liouville operator in terms of multipliers and weighted characteristics of the coefficients have been proved.

4. Estimates of the best linear approximations of solutions to singular differential equations in terms of special maximal operators constructed on a basis of intervals of variable length have been obtained.

5. Estimates of linear widths of solution sets of singular differential equations in weighted functional spaces have been obtained.

Theoretical and practical significance of the research. The results obtained are theoretical in nature and contribute to the development of the theory of singular differential operators and functional analysis. The methods developed and results obtained can be used in the study of differential operators and equations with singular coefficients, in the spectral theory of operators, problems of numerical analysis and approximation theory, as well as in special courses on functional analysis and the theory of differential equations.

Main results submitted for defence.

1. Conditions for the existence of closed and invertible extensions of singular second-order differential operators with non-smooth and sign-alternating coefficients.

2. Coercive estimates for singular differential operators in weighted Sobolev spaces, Lebesgue spaces, and multiplier spaces.

3. Theorems on the separability of the Sturm–Liouville operator in weighted functional spaces.

4. Estimates of the best linear approximations of solutions to singular differential equations.

5. Estimates of linear widths of solution sets of differential equations.

Approbation of the work. The main results of the dissertation were presented and discussed at the following conferences and seminars:

- International scientific conference "Theoretical and Applied Issues of Mathematics, Mechanics and Computer Science" (Karaganda, 2019);
- International scientific conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019);
- International scientific conference dedicated to the 10th Anniversary of the Eurasian Mathematical Journal (Nur-Sultan, 2019);
- International scientific conference "Problems of Modern Fundamental and Applied Mathematics" (Nur-Sultan, 2021);
- Scientific seminar "Seminario Dottorato del Dipartimento di Matematica" under the supervision of Professors Corrado Marastoni and Tiziano Vargiolu (Padova, 2021);

- Scientific seminar "Functional Analysis and Its Applications" under the supervision of Academicians of the National Academy of Sciences of the Republic of Kazakhstan M. Otelbaev, R.O. Oinarov, Professors E.D. Nursultanov, K.N. Ospanov (Astana, 2025);
- Scientific seminar "Modern Problems of Mathematics and Computer Science" under the supervision of Doctor of Physical and Mathematical Sciences, Professor E.D. Nursultanov (Astana, 2025).

Publications. The main material presented in the dissertation was published in:

1. On the separation property of the Sturm-Liouville operator in weighted spaces of multiplicators // Journal of Mathematical Sciences. – 2019. – Vol. 241, No. 5. – P. 596–604.
2. On an invertible extension of a non-self-adjoint singular differential operator on the axis // Differential Equations. – 2021. – Vol. 57, No. 10. – P. 1431–1435.
3. On approximation of one singular differential equation on the axis //Bulletin of the Karaganda University. Mathematics Series. – 2022. – No. 4(108). – P. 86–97.
4. On a coercive estimate in multiplier spaces //Bulletin of Abai KazNPU, series "Physical and Mathematical Sciences". – 2023. – No. 2(82). – P. 22–29.
5. On some weighted estimates and their applications in the theory of differential operators // International Scientific Conference "Theoretical and Applied Issues of Mathematics, Mechanics and Computer Science" (Karaganda, 2019. – P. 33).
6. Coercive estimates for a differential operator on the axis in multiplier spaces //"Actual Problems of Analysis, Differential Equations and Algebra": International Conference dedicated to the 10th anniversary of the Eurasian Mathematical Journal (Nur-Sultan, 2019. – P. 40–42).
7. On an estimate for solutions of a homogeneous second-order differential equation //International Scientific-Practical Conference "Problems of Modern Fundamental and Applied Mathematics" (Nur-Sultan, 2021. – P. 161–163).

Structure and scope of the dissertation. The work is 92 pages in length and consists of an introduction, 3 sections, a conclusion, and a list of references.

The first section investigates a singular second-order differential operator on the axis with locally summable coefficients admitting non-smoothness, unboundedness, and unbounded sign alternation. To solve the stated problems, special length functions are introduced, the properties of admissible weighted pairs are studied, and local relationships between the coefficients of the operator on characteristic intervals are established. This section obtains functional characteristics of weighted Sobolev spaces, proves theorems on the existence of a closed extension of the operator under study, and derives sufficient conditions for its invertibility. Coercive estimates are established relating the norm of the solution to the norm of the right-hand side of the equation.

The same section considers the problem of the best linear approximation of the solution set of a singular differential equation. For the corresponding solution classes, estimates of approximation numbers and estimates of the best linear approximations are obtained in terms of special maximal operators constructed on a basis of intervals of variable length. Examples are constructed illustrating the application of the results obtained.

The second section investigates a second-order differential operator with coefficients from the Triebel class and considers questions of solvability of the corresponding differential equations in weighted spaces. For the class of operators under study, functional spaces are constructed, their main properties are described, and characteristics of the domains of definition of the closed operators are obtained. Theorems on the existence and boundedness of inverse operators are proved. As in the first section,

the problem of best linear approximations is studied for solution sets of the corresponding differential equations. Examples illustrating the application of the results to specific classes of singular differential equations are constructed.

The third section obtains coercive estimates for singular second-order differential operators both in classical functional spaces and in spaces of pointwise multipliers. Questions of operator separability and functional characteristics of solutions to the corresponding differential equations are studied. For a wide class of weighted Sobolev and Lebesgue spaces, coercive inequalities in multiplier spaces are proved. Sufficient conditions are obtained under which solutions of the equations under study are multipliers between weighted functional spaces. Special attention is paid to the Sturm–Liouville operator with a singular potential. For this operator, separability theorems are proved in terms of weighted characteristics and multipliers. New separability criteria are obtained that substantially expand the class of potentials for which the operator possesses this property. Examples demonstrating the application of the theoretical results obtained are constructed.

The doctoral candidate's contribution to each publication. In all published works, the formulation of problems and the choice of research methods were carried out jointly with scientific advisors. The doctoral candidate independently obtained the main and auxiliary results, conducted the proofs of theorems, and completed the presentation of the research results.

Connection of this work with other research projects. The topic of the dissertation research corresponds to the priority development direction "Scientific Research in the Natural Sciences", specialized scientific direction "Fundamental and Applied Research in Mathematics and Mechanics". Some results of the dissertation were included in the interim report for 2021 under project AP08856104 "Differential Operators with Singular Coefficients. Approximative Characteristics. Spectrum. Inverse Problems" and for 2019 under project AP05133397 "Qualitative and Quantitative Characteristics of Singular Differential Operators. Multipliers in the Theory of Differential Operators".

Number of references used: 58.

Keywords: singular differential operators, weighted estimates, coercive estimates, separability of operators, Sturm–Liouville operator, Sobolev spaces, Lebesgue spaces, multiplier spaces, closed extensions of operators, singular coefficients, locally summable coefficients.